July 1996, rev. Nov. 1996 LMU-TPW 96-17

On Bose-Fermi Statistics, Quantum Group Symmetry, and Second Quantization[1] [2]

Gaetano Fiore

Sektion Physik der Ludwig-Maximilians-Universität München
Theoretische Physik — Lehrstuhl Professor Wess
Theresienstraße 37, 80333 München
Federal Republic of Germany
e-mail: Gaetano.Fiore @ physik.uni-muenchen.de

Abstract

Can one represent quantum group covariant q-commuting "creators, annihilators" A_i^+, A^j as operators acting on *standard* bosonic/fermionic Fock spaces? We briefly address this general problem and show that the answer is positive (at least) in some simplest cases.

[1] Talk presented at the XXI International Colloquium on Group Theoretical Methods in Physics (Group21), 15 - 20 July 1996, Goslar, Germany. To appear in the proceedings of the Quantum Group volume of Group21, Eds. H.-D. Doebner and V.K. Dobrev.

[2] Work supported by the European Community, through the European Commission, Dir. Gen. for Science, Research and Development, providing the TMR grant ERBFMBICT960921.

1 Introduction

In recent years the idea of Quantum Field Theories (QFT) endowed with Quantum Group [1] symmetries has attracted considerable interest and has been investigated especially in 2D field theories, in connection with socalled anyonic statistics (when the deformation parameter q is a root of unity). Its application to QFT in higher (*e.g.* 3+1) space-time dimensions relies, among other things, on the condition that Bose and Fermi statistics are compatible with quantum group-symmetry transformations, at other (in particular real) values of q. The latter issue in fact involves two different problems, one in first quantized quantum mechanics and the other in QFT.

The first problem essentially is whether a Hilbert space can carry both a completely (anti)symmetric representation of the symmetric group S_n (so that it can describe the states of n bosons/fermions) and of a quasitriangular non-cocommutative $*$-Hopf algebra H. Contrary to a quite widespread prejudice, we showed in Ref. [2] that this is possible whenever H can be obtained from the universal enveloping Ug of a Lie algebra g by a unitary "Drinfel'd twist" $\mathcal{F}$ [3, 4]. Only, we need to describe the system of n bosons/fermions in an unusual picture, that is related to the standard one [involving (anti)symmetric wavefunctions and symmetric operators] by a unitary transformation $F_{12...n}$ *not* symmetric under tensor factor permutations; $F_{12...n}$ is derived from $\mathcal{F}$. The relevant point here is that even in this scheme second quantization naturally leads [5] to creation and annihilation operators A_i^{+c}, A_c^j satisfying the *canonical* (anti)commutation relations (CCR), and to the standard bosonic/fermionic Fock space representations, exactly as in the standard treatment of second quantization.

The second problem, which we briefly address here, is whether nonetheless one can represent, as operators acting on standard bosonic/fermionic spaces, algebraic objects A_i^+, A^j: (1) transforming as conjugate tensors under the action of H; (2) satisfying the *quantum*, *i.e.* H-covariant, commutation relations (QCR) [6, 7]. We report here results (proved in Ref. [8]) which allow a positive answer to requirement (1), under the same assumption as above, and a positive answer to requirement (2): a) in the simpler case that H is triangular; b) in the particular case that $H = U_q su(2)$ and A_i^{+c}, A_c^j belong to the fundamental representation of $su(2)$[3].

[3]We expect the same result also for other compact $U_q\mathbf{g}$ (at least when ρ is a fundamental representation), but so far could not prove it, due to the limited knowledge about their $\mathcal{F}$.

We look for a realization of A_i^+, A^j in the form of formal power series in the A_i^{+c}, A_c^j[4]. Using $\mathcal{F}$, in sect. 3 we determine a class of candidates for A_i^+, A^j fulfilling requirement (1). In Sect 4 we show how to pick out of this class a particular set satisfying requirement (2) under one of the assumptions a), b). These A_i^+, A^j turn out to be well-defined operators on the bosonic/fermion Fock space. In sect. 5 we briefly comment on the possible application of our results to QFT.

2 Preliminaries and notation

2.1 Twisting groups into quantum groups

Let $(U\mathbf{g}\,, m, \Delta_c, \varepsilon, S_c)$ be the cocommutative Hopf algebra associated with the universal enveloping (UE) algebra $U\mathbf{g}$ of a Lie algebra $\mathbf{g}$. $m, \Delta_c, \varepsilon, S_c$ denote the multiplication comultiplication, counit and antipode respectively; we will often drop the symbol m: $m(a \otimes b) \equiv ab$.

Let $\mathcal{F} \in U\mathbf{g}\,[[\hbar]] \otimes U\mathbf{g}\,[[\hbar]]$ (we will write $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$, in a Sweedler's notation with *upper* indices) be a twist, *i.e.* an invertible element satisfying the relations

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)\mathcal{F} \tag{2.1}$$

and $\mathcal{F}|_{\hbar=0} = \mathbf{1} \otimes \mathbf{1}$ ($\hbar \in \mathbf{C}$ is the 'deformation parameter'). It is well known [3] that if $\mathcal{F}$ also satisfies the relation

$$(\mathcal{F} \otimes \mathbf{1})[(\Delta_c \otimes \mathrm{id})(\mathcal{F})] = (\mathbf{1} \otimes \mathcal{F})[(\mathrm{id} \otimes \Delta_c)(\mathcal{F}), \tag{2.2}$$

then one can construct a triangular non-cocommutative Hopf algebra $H = (U\mathbf{g}\,[[\hbar]], m, \Delta, \varepsilon, S, \mathcal{R}\,)$ having the same algebra structure $(U\mathbf{g}\,[[\hbar]], m)$, the same counit ε, comultiplication and antipode defined by

$$\Delta(a) = \mathcal{F}\Delta_c(a)\mathcal{F}^{-1}, \qquad\qquad S(a) = \gamma^{-1}S_c(a)\gamma \tag{2.3}$$

(where $\gamma^{-1} := \mathcal{F}^{(1)} \cdot S_c\mathcal{F}^{(2)}$), and (triangular) universal R-matrix $\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}$ ($\mathcal{F}_{21} := \mathcal{F}^{(2)} \otimes \mathcal{F}^{(1)}$). Condition (2.2) ensures that Δ is coassociative as Δ_c.

Examples of $\mathcal{F}$'s satisfying conditions (2.2), (2.1) are provided *e.g.* by the so-called 'Reshetikhin twists' $\mathcal{F} := e^{\hbar\omega_{ij}h_i \wedge h_j}$, where $\{h_i\}$ is a basis in the Cartan subalgebra of $\mathbf{g}$ and $\omega_{ij} \in \mathbf{C}$.

[4] A_i^{+c}, A_c^j tranform as tensors under the action of the classical group, *not* of the quantum group.

A similar result holds for genuine quantum groups. A well-known theorem by Drinfel'd [4] essentially proves, for any *quasitriangular* deformation $H = (U_q\mathbf{g}\ , m, \Delta, \varepsilon, S)$ [1, 9] of $U\mathbf{g}$, with $\mathbf{g}$ simple belonging to the classical A,B,C,D series, the existence of an invertible $\mathcal{F}$ satisfying condition (2.1) such that H can be obtained from $U\mathbf{g}$ through formulae (2.3) as well, after identifying $U_q\mathbf{g}$ with the isomorphical algebra $U\mathbf{g}\,[[\hbar]]$, $\hbar = \ln q$. This $\mathcal{F}$ does not satisfy condition (2.2), however the (nontrivial) coassociator $\phi := \mathcal{F}^{-1}_{12,3}\mathcal{F}^{-1}_{12}\mathcal{F}_{23}\mathcal{F}_{1,23} \in U\mathbf{g}^{\otimes^3}$ still commutes with $\Delta^{(2)}_c(U\mathbf{g}\,)$, thus explaining why Δ is coassociative in this case, too. The corresponding universal (quasitriangular) R-matrix $\mathcal{R}$ is related to $\mathcal{F}$ by $\mathcal{R} := \mathcal{F}_{21}q^{\frac{t}{2}}\mathcal{F}^{-1}$, where $t := \Delta_c(\mathcal{C}) - \mathbf{1}\otimes\mathcal{C} - \mathcal{C}\otimes\mathbf{1}$ is the canonical invariant element in $U\mathbf{g}\otimes U\mathbf{g}$ ($\mathcal{C}$ is the quadratic Casimir).

In defining ϕ we have used a 'tensor notation' which will be repeatedly employed in the sequel. According to it, eq. (2.2) can be rephrased as $\mathcal{F}_{12}\mathcal{F}_{12,3} = \mathcal{F}_{23}\mathcal{F}_{1,23}$; the comma separates the tensor factors *not* stemming from the coproduct. On the other hand, we will use unbarred and barred indices to distinguish Δ from Δ_c in Sweedler's notation: $\Delta_c(x) \equiv x_{(1)}\otimes x_{(2)}$, $\Delta(x) \equiv x_{(\bar{1})}\otimes x_{(\bar{2})}$.

2.2 Classically covariant creators and annihilators

Let $\mathcal{A}_\pm$ be the unital algebra generated by $\mathbf{1}_\mathcal{A}$ and elements $\{A^{+c}_i\}_{i\in I}$ and $\{A^j_c\}_{j\in I}$ satisfying the (anti)commutation relations

$$
\begin{aligned}
[A^i_c\,,\,A^j_c]_\pm &= 0\\
[A^{+c}_i\,,\,A^{+c}_j]_\pm &= 0\\
[A^i_c\,,\,A^{+c}_j]_\pm &= \delta^i_j\mathbf{1}_\mathcal{A}
\end{aligned}
\tag{2.4}
$$

(the $\pm$ sign denotes commutators and anticommutators respectively), belonging respectively to some representation ρ and to its contragradient $\rho^\vee_c = \rho^T\circ S_c$ of $U\mathbf{g}$ (^{T} is the transpose):

$$
x\overset{c}{\triangleright}A^{+c}_i = \rho(x)^l_iA^{+c}_l \qquad x\overset{c}{\triangleright}A^i_c = \rho(S_cx)^i_lA^l_c. \qquad x\in U\mathbf{g}\,,\quad \rho(x)^i_j\in\mathbf{C}. \tag{2.5}
$$

Equivalently, one says that A^{+c}_i, A^i_c are "covariant", or "tensors", under $\overset{c}{\triangleright}$.

$\mathcal{A}_\pm$ is a (left) module of $(U\mathbf{g}\,,\overset{c}{\triangleright})$, if the action $\overset{c}{\triangleright}$ is extended on the whole $\mathcal{A}_\pm$ by means of the (cocommutative) coproduct:

$$
x\overset{c}{\triangleright}(ab) = (x_{(1)}\overset{c}{\triangleright}a)(x_{(2)}\overset{c}{\triangleright}b). \tag{2.6}
$$

Setting

$$\sigma(X) := \rho(X)^i_j A^{+c}_i A^j_c \tag{2.7}$$

for all $X \in \mathbf{g}$, one finds that $\sigma : \mathbf{g} \to \mathcal{A}_\pm$ is a Lie algebra homomorphism, so that σ can be extended to all of $U\mathbf{g}\,[[\hbar]]$ as an algebra homomorphism $\sigma : U\mathbf{g}\,[[\hbar]] \to \mathcal{A}_\pm[[\hbar]]$; on the unit element we set $\sigma(\mathbf{1}_{U\mathbf{g}}) := \mathbf{1}_{\mathcal{A}}$. $\sigma(X)$ commutes with the 'number of particles' $N^c := A^{+c}_i A^i_c$. σ can be seen as the generalization of the Jordan-Schwinger realization of $su(2)$,

$$\sigma(j_+) = A^{+c}_\uparrow A^\downarrow_c, \qquad \sigma(j_-) = A^{+c}_\downarrow A^\uparrow_c, \qquad \sigma(j_0) = \frac{1}{2}(A^{+c}_\uparrow A^\uparrow_c - A^{+c}_\downarrow A^\downarrow_c). \tag{2.8}$$

Lemma 1 *The (left) action* $\overset{c}{\triangleright} : U\mathbf{g} \times \mathcal{A}_\pm \to \mathcal{A}_\pm$ *can be realized in an 'adjoint-like' way:*

$$x \overset{c}{\triangleright} a = \sigma(x_{(1)})\, a\, \sigma(S_c x_{(2)}), \qquad x \in U\mathbf{g}, \qquad a \in \mathcal{A}_\pm. \tag{2.9}$$

3 Deforming maps to quantum group covariant creators and annihilators

On the other hand, it is straightforward to check that the definition

$$x \triangleright a := \sigma(x_{(\bar{1})}) a \sigma(S x_{(\bar{2})}) \tag{3.10}$$

allows to realize the "quantum" (left) action of H on the left module $\mathcal{A}_\pm[[\hbar]]$, *i.e.* that $(xy) \triangleright a = x \triangleright (y \triangleright a)$ and $x \triangleright (ab) = (x_{(\bar{1})} \triangleright a)(x_{(\bar{2})} \triangleright b)$ $\forall x, y \in H$, $a, b \in \mathcal{A}_\pm[[\hbar]]$.

However, A^{+c}_i, A^j_c are *not* covariant w.r.t. to $\triangleright$. Are there covariant objects $A^+_i, A^j \in \mathcal{A}_\pm$ (going to A^{+c}_i, A^j_c in the limit $\hbar \to 0$)? The answer comes from

Proposition 1 *[8] For any invertible* $\mathbf{g}$ *-invariant [i.e. , commuting with* $\Delta_c(U\mathbf{g})$*] elements* $T, T' \in U\mathbf{g}\,[[\hbar]] \otimes U\mathbf{g}\,[[\hbar]]$ *the elements*

$$\begin{aligned} A^+_i &:= \sigma(Q^{(1)}) A^{+c}_i \sigma(S_c Q^{(2)} \gamma) \in \mathcal{A}_\pm[[\hbar]] \\ A^i &:= \sigma(\gamma' S_c Q'^{(2)}) A^i_c \sigma(Q'^{(1)}) \in \mathcal{A}_\pm[[\hbar]] \end{aligned} \tag{3.11}$$

($Q := \mathcal{F}T$ $Q' := \mathcal{F}^{-1}T'$, $\gamma' := \left[S_c \mathcal{F}^{-1(2)} \cdot \mathcal{F}^{-1(1)}\right]^{-1}$*) are "covariant" under* $\triangleright$*, more precisely belong respectively to the irreducible representations* ρ *and to its quantum contragredient one* $\rho^\vee = \rho^T \circ S$ *of* H *acting through* $\triangleright$*:*

$$x \triangleright A^+_i = \rho(x)^l_i A^+_l \qquad x \triangleright A^i = \rho(Sx)^i_m A^m. \tag{3.12}$$

Remark 1. If H is a $*$-Hopf algebra, ρ, $\mathcal{F}$ are unitary[5] and $\dagger$ is an involution in $\mathcal{A}_\pm$, then $\gamma' = \gamma^*$ and

$$(A^i_c)^\dagger = A^{+c}_i \qquad \Rightarrow \qquad \sigma \circ * = \dagger \circ \sigma, \qquad (A^i)^\dagger = A^+_i. \tag{3.13}$$

Remark 2. Let ${\mathcal{A}_\pm}^{c,inv}, {\mathcal{A}_\pm}^{inv}$ be the subalgebras of $\mathcal{A}_\pm[[\hbar]]$ invariant under $\stackrel{c}{\triangleright}, \triangleright$ (*i.e.* $I \in {\mathcal{A}_\pm}^{c,inv}$ iff $x \triangleright I = \varepsilon(x) I$, $I \in {\mathcal{A}_\pm}^{inv}$ iff $x \triangleright I = \varepsilon(x) I$). It is not difficult to prove that ${\mathcal{A}_\pm}^{c,inv} = {\mathcal{A}_\pm}^{inv}$. An element $I \in \mathcal{A}_\pm[[\hbar]]$ can be expressed as a function of A^i, A^+_j or of A^i_c, A^{+c}_j, $I = f(A^i, A^+_j) = f_c(A^i_c, A^{+c}_j)$. We will prove elsewhere that $f = f_c$ in the triangular case, but not in the genuine quasitriangular one. In the latter case, to a polynomial f (resp. f_c) there corresponds a highly non-polynomial (tipically a trascendental function) f_c (resp. f); so the change of generators $A^i, A^+_j \leftrightarrow A^i_c, A^{+c}_j$ can be used to simplify the functional dependence of I (what might turn useful for practical purposes, *e.g.* to solve the dynamics associated to some Hamiltonian I).

Remark 3. A^i, A^+_j are well-defined as operators on the bosonic/fermionic Fock spaces, at least for small $\hbar$ [assuming that the tensors T, T' are also of the form $\mathbf{1} \otimes \mathbf{1} + O(\hbar)$]; correspondingly, the transformation $A^i_c, A^{+c}_j \to A^i, A^+_j$ is invertible.

4 Fulfilment of the "quantum" commutation relations

Theorem 1 *[8] If the noncommutative Hopf algebra H is* **triangular** *[i.e. the twist $\mathcal{F}$ satisfies equation (2.1)], then, setting $T \equiv \mathbf{1} \otimes \mathbf{1} \equiv T'$ in eq. (3.11), A^i, A^+_j close the quadratic commutation relations*

$$\begin{aligned} A^i A^+_j &= \delta^i_j \mathbf{1}_{\mathcal{A}} \pm R^{ui}_{jv} A^+_u A^v, & (4.14)\\ A^i A^j &= \pm R^{ij}_{vu} A^u A^v & (4.15)\\ A^+_i A^+_j &= \pm R^{vu}_{ij} A^+_u A^+_v & (4.16)\end{aligned}$$

where R is the (numerical) quantum R-matrix of $U\mathbf{g}$ in the representation ρ,

$$R^{ij}_{hk} := [(\rho \otimes \rho)(\mathcal{R})]^{ij}_{hk}. \tag{4.17}$$

[5] One can always choose $\mathcal{F}$ unitary if $\mathbf{g}$ is compact [10].

Theorem 2 *[8]* [6] *If $g = su(2)$ and $\rho \equiv$ fundamental representation, it is possible to determine T, T' such that the elements $A^i, A^+_j \in \mathcal{A}_\pm[[\hbar]]$ $(i,j = \uparrow,\downarrow)$ defined in formulae (3.11) are covariant under $U_q su(2)$ and satisfy the $U_q su(2)$-covariant quadratic QCR [6, 11, 7]*

$$\begin{aligned} A^i A^+_j &= \mathbf{1}_{\mathcal{A}} \delta^i_j \pm q^{\pm 1} R^{ui}_{jv} A^+_u A^v, && (4.18)\\ A^i A^j &= \pm q^{\mp 1} R^{ij}_{vu} A^u A^v && (4.19)\\ A^+_i A^+_j &= \pm q^{\mp 1} R^{vu}_{ij} A^+_u A^+_v, && (4.20) \end{aligned}$$

where R is the R-matrix of $U_q sl(2)$. Moreover, $(A^i)^\dagger = A^+_i$ for the compact section $U_q su(2)$ $(q \in \mathbf{R})$. With this choice of T, T', A^+_i, A^j explicitly read, in the bosonic case,

$$\begin{array}{lllllll} A^+_\uparrow &=& \sqrt{\frac{(N^\uparrow_c)_{q^2}}{N^\uparrow_c}}\, q^{N^\downarrow_c} A^{+c}_\uparrow & \qquad & A^+_\downarrow &=& \sqrt{\frac{(N^\downarrow_c)_{q^2}}{N^\downarrow_c}}\, A^{+c}_\downarrow \\ A^\uparrow &=& A^\uparrow_c \sqrt{\frac{(N^\uparrow_c)_{q^2}}{N^\uparrow_c}}\, q^{N^\downarrow_c} & & A^\downarrow &=& A^\downarrow_c \sqrt{\frac{(N^\downarrow_c)_{q^2}}{N^\downarrow_c}}, \end{array} \qquad (4.21)$$

and in the 'fermionic' one

$$\begin{array}{lllllll} A^+_\uparrow &=& q^{-N^\downarrow_c} A^{+c}_\uparrow & \qquad & A^+_\downarrow &=& A^{+c}_\downarrow \\ A^\uparrow &=& A^\uparrow_c q^{-N^\downarrow_c} & & A^\downarrow &=& A^\downarrow_c, \end{array} \qquad (4.22)$$

where $N^\uparrow_c := A^{+c}_\uparrow A^\uparrow_c$, $\quad N^\downarrow_c := A^{+c}_\downarrow A^\downarrow_c$, $\quad (x)_{q^2} := \frac{q^{2x}-1}{q^2-1}$.

5 Application to QFT

If the representation ρ is reducible the algebra homomorphism σ defined in eq. (2.7) contains a sum over all the irreducible components. In the case of a QFT, the generators of the Heisenberg algebra are fields $\phi^{i_\alpha}_c(\vec{x})$ and there conjugate momenta $\pi^c_{i_\alpha}(\vec{x})$ (satisfying the commutation relations $[\phi^{i_\alpha}_c(\vec{x}), \pi^c_{j_\beta}(\vec{x}')]_\pm = i\delta^{(3)}(\vec{x}-\vec{x}')\delta^\alpha_\beta \delta^{i_\alpha}_{j_\beta}$), *i.e.* depend also on a continuous space index, so the sum entails also an integral:

$$\sigma(X^a) = i \int d^3x \quad \sum_\alpha \rho_\alpha(X^a)^{i_\alpha}_{j_\alpha} \pi^c_{i_\alpha}(x) \phi^{j_\alpha}_c(x) \qquad\qquad X^a \in \mathbf{g}\,; \qquad (5.23)$$

the index α enumerates all the kinds of different fields (*i.e.* particles) of the theory. At the RHS we recognize the charge Q^a associated to the generator $X^a \in \mathbf{g}$.

[6] In the proof of theorem 6 we made [8] essential use of the $U\mathbf{g}\,[[\hbar]]$-valued 2×2 matrix $(\rho\otimes\mathrm{id})\mathcal{F}$ found in Ref. [12].

The operators $\pi_{i_\alpha}(x)\phi^{j_\alpha}(x)$ which are obtained from the canonical ones $\pi^c_{i_\alpha}(x)$, $\phi_c^{j_\alpha}(x)$ through the transformation (3.11), are well-defined (nonlocal) composite operators on the Fock space generated by $\phi_c^{i_\alpha}(\vec{x})$; they act as $\pi^c_{i_\alpha}(x), \phi_c^{j_\alpha}(x)$ "dressed" in a peculiar way by all the fields considered in the theory. In the case of a triangular Hopf algebra H, theorem 1 implies that $\pi_{i_\alpha}(x), \phi^{j_\alpha}(x)$ satisfy the quadratic commutation relations

$$\phi^{i_\alpha}(\vec{x})\pi_{j_\beta}(\vec{x}') = i\delta^{(3)}(\vec{x}-\vec{x}')\delta^\alpha_\beta\delta^{i_\alpha}_{j_\beta} \pm R^{l_\beta i_\alpha}_{j_\beta m_\alpha}\pi_{l_\beta}(\vec{x}')\phi^{m_\alpha}(\vec{x}) \tag{5.24}$$

$$\phi^{i_\alpha}(\vec{x})\phi^{j_\beta}(\vec{x}') = \pm R^{i_\alpha j_\beta}_{m_\alpha l_\beta}\phi^{l_\beta}(\vec{x}')\phi^{m_\alpha}(\vec{x}) \tag{5.25}$$

$$\pi_{i_\alpha}(\vec{x})\pi_{j_\beta}(\vec{x}') = \pm R^{m_\alpha l_\beta}_{i_\alpha j_\beta}\pi_{l_\beta}(\vec{x}')\pi_{m_\alpha}(\vec{x}) \tag{5.26}$$

where $R^{i_\alpha j_\beta}_{m_\alpha l_\beta} = [(\rho_\alpha \otimes \rho_\beta)(\mathcal{R})]^{i_\alpha j_\beta}_{m_\alpha l_\beta}$. Because of remark 2 in sect. 3, in this case an invariant action $\mathcal{S}$ has the same functional dependence on $\pi_{i_\alpha}(x), \phi^{j_\alpha}(x)$ as on $\pi^c_{i_\alpha}(x), \phi_c^{j_\alpha}(x)$.

In the quasitriangular case $H = U_q sl(2)$ theorem 6 is not applicable, because in its present form its validity is restricted only to the fundamental ρ (so the operators $\pi_{i_\alpha}(x), \phi^{j_\alpha}(x)$ do not satisfy quadratic commutation relations). Whether some generalization of this theorem to arbitrary ρ exists and these ideas can be applied to QFT also for quasitriangular H's, is presently only matter of speculations.

Acknowledgments

I am grateful to Prof.'s V. K. Dobrev and H.-D. Doebner for their invitation to Group21. It is a pleasure to thank J. Wess for his support and for the hospitality at his Institute.

References

[1] V. G. Drinfeld, *Quantum Groups*, Proceedings of the International Congress of Mathematicians, Berkeley 1986, Vol. 1, 798.

[2] G. Fiore and P. Schupp, *Identical Particles and Quantum Symmetries*, Nucl. Phys. **B470** (1996), 211.

[3] V. G. Drinfeld, Doklady AN SSSR **273** (1983), 531.

[4] V. G. Drinfeld, *Quasi Hopf Algebras*, Leningrad Math. J. **1** (1990), 1419.

[5] G. Fiore and P. Schupp, *Statistics and Quantum Group Symmetries*, Preprint LMU-TPW-96-06 ed e-print hep-th/9605133 to appear in *"Quantum Groups and Quantum Spaces", Banach Center Publications 40 (1997), Inst. of Mathematics, Polish Academy of Sciences, Warszawa*, P. Budzyski, W. Pusz, S. Zakrweski (editors).

[6] W. Pusz, S. L. Woronowicz, *Twisted Second Quantization*, Reports on Mathematical Physics **27** (1989), 231.

[7] J. Wess and B. Zumino, *Covariant Differential Calculus on the Quantum Hyperplane* Nucl. Phys. Proc. Suppl. **18B** (1991), 302.

[8] G. Fiore, *Deforming Maps from Group to Quantum Group Covariant Creation & Annihilation Operators*, Preprint LMU-TPW 96-20, and e-print q-alg/9610005.

[9] L. D. Faddeev, N. Y. Reshetikhin and L. A. Takhtajan, *Quantization of Lie Groups and Lie Algebras*, Algebra i Analysis, **1** (1989), 178; translation: Leningrad Math. J. **1** (1990), 193.

[10] B. Jurco, *More on Quantum Groups from the Quantization Point of View*, Commun. Math. Phys. **166** (1994), 63.

[11] W. Pusz, *Twisted Canonical Anticommutation Relations*, Reports on Mathematical Physics **27** (1989), 349.

[12] T. L. Curtright, G. I. Ghandour, C. K. Zachos, J. Math. Phys. **32** (1991), 676.